Outer space and Automorphisms of free groups

LECTURE 3

In lecture 1 we compared $\mathrm{Out}(F_n)$ with $GL_n\mathbb{Z}$ and with surface mapping class groups $\mathrm{Mod}(S_{g,s})$

These groups are studied via their actions on symmetric spaces $(SO(n)\backslash SL_n\mathbb{R})$ and Teichmüller spaces $(\mathcal{T}_{g,s})$

Outer space CV_n is the analog for $\mathrm{Out}(F_n)$

Useful properties of these spaces:

- contractible
- finite-dimensional
- action is proper (stabilizers are finite)

We've modeled F_n by graphs and by doubled handlebodies. You can (and we will) define CV_n in terms of either of these

But the quickest definition relies on a third characterization of free groups:

$$\Gamma \text{ is free} \iff \Gamma \text{ acts freely on a tree}$$

Tree = 1-dimensional simplicial complex which is connected and 1-connected

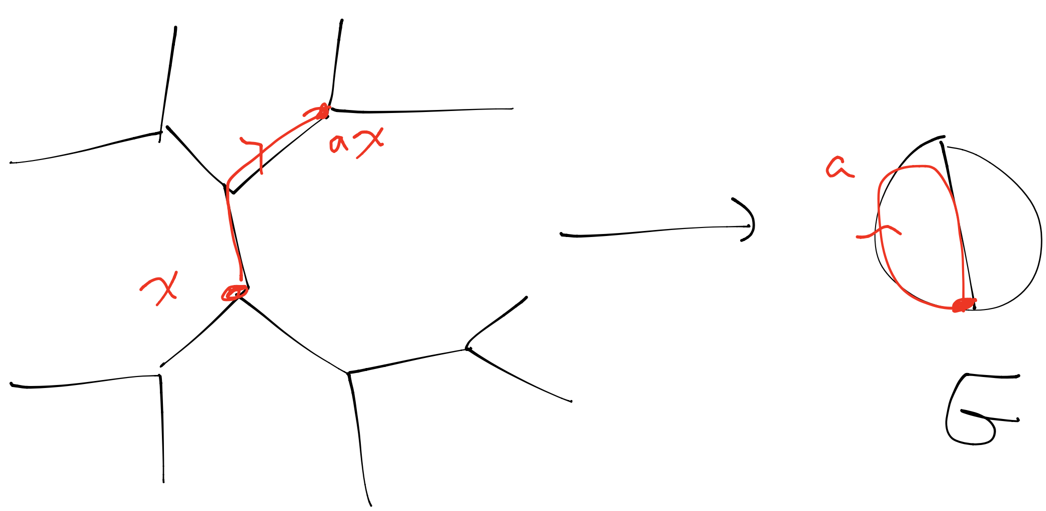
action is by simplicial automorphisms

(vertices $\to$ vertices, edges $\to$ edges)

An action is free if every $g \in \Gamma$ moves every point of T

eg G = graph, $\Gamma = \pi_1(G)$, take $T = \tilde{G}$,

action by deck transformations is free, simplicial



We want to make a space of free actions of F_n on trees

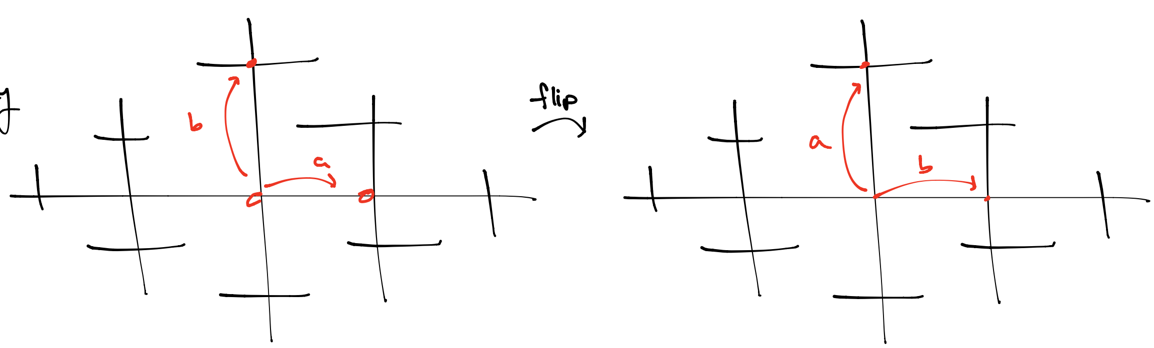
To cut down the size ($\leftrightarrow$ dimension) only consider minimal actions: no invariant subtrees

To make a continuous space, put a metric on T, (each edge isometric to interval in $\mathbb{R}$), use actions by isometries

Two actions $F_n \xrightarrow{\rho} \mathrm{Isom}(T)$ and $F_n \xrightarrow{\rho'} \mathrm{Isom}(T')$ are equivalent if there is an isometry $T \to T'$ which commutes with the actions:

$$\begin{tikzcd} T \arrow[r] \arrow[d, "\rho(g)"'] & T' \arrow[d, "\rho'(g)"] \\ T \arrow[r] & T' \end{tikzcd}$$

eg



Definition #1 CV_n is the space of equiv. classes of free minimal actions of F_n on metric simplicial trees

Topology = equivariant Gromov-Hausdorff topology

GH topology on metric spaces: T' is in ε-nbd of T if every finite set X in T has a matching X' in T' s.t. corresponding distances are within ε.

Equivariant GH topology: takes action into account need finite $X \subset T$ and finite $A \subset F_n$ to decide whether $\rho: F_n \to \mathrm{Isom}(T)$ and $\rho': F_n \to \mathrm{Isom}(T')$ are close

Formally: A neighborhood basis for the equivariant Gromov-Hausdorff topology = sets $V_\rho(X, A, \varepsilon)$
$X \subset T$, $A \subset F_n$ finite, $\varepsilon > 0$

$\rho' \in V_\rho(X, A, \varepsilon)$ if $\exists\, X' \subset T'$, bijection $X \leftrightarrow X'$
s.t. $|d(x, gy) - d(x', gy')| < \varepsilon \quad \forall\, x, y \in X,\ g \in A$

<u>Action of Out(F_n)</u>: Lift $\varphi \in \mathrm{Out}\, F_n$ to $\hat{\varphi} \in \mathrm{Aut}(F_n)$

$$\begin{tikzcd} F_n \arrow[r, "\rho"] & \mathrm{Isom}(T) \\ F_n \arrow[u, "\hat{\varphi}"] \arrow[ur, "\rho\circ\hat{\varphi}"'] & \end{tikzcd}$$

$$\rho \cdot \varphi = \rho \circ \hat{\varphi}$$

<u>same</u> T, <u>different</u> action

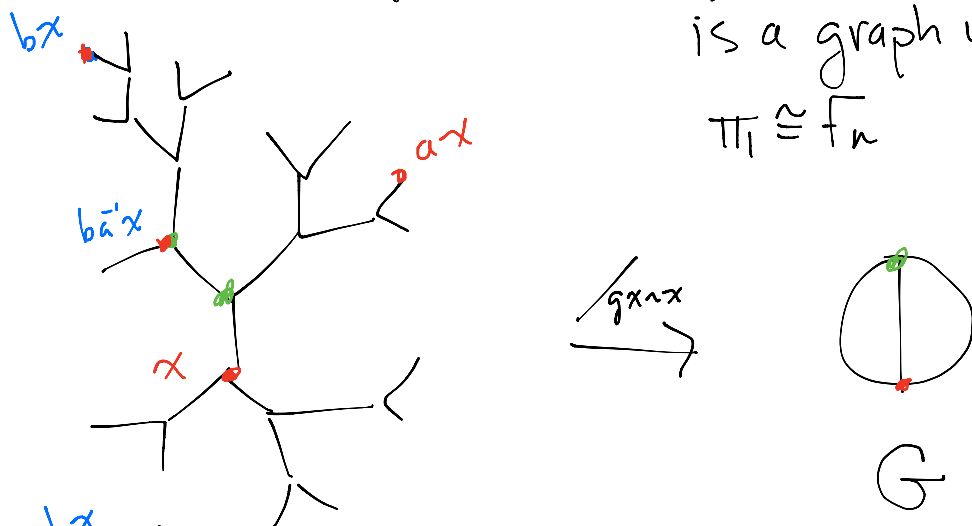
<u>Exercise</u>: Why is this an action of Out F_n? (Show that inner automorphisms act trivially.)

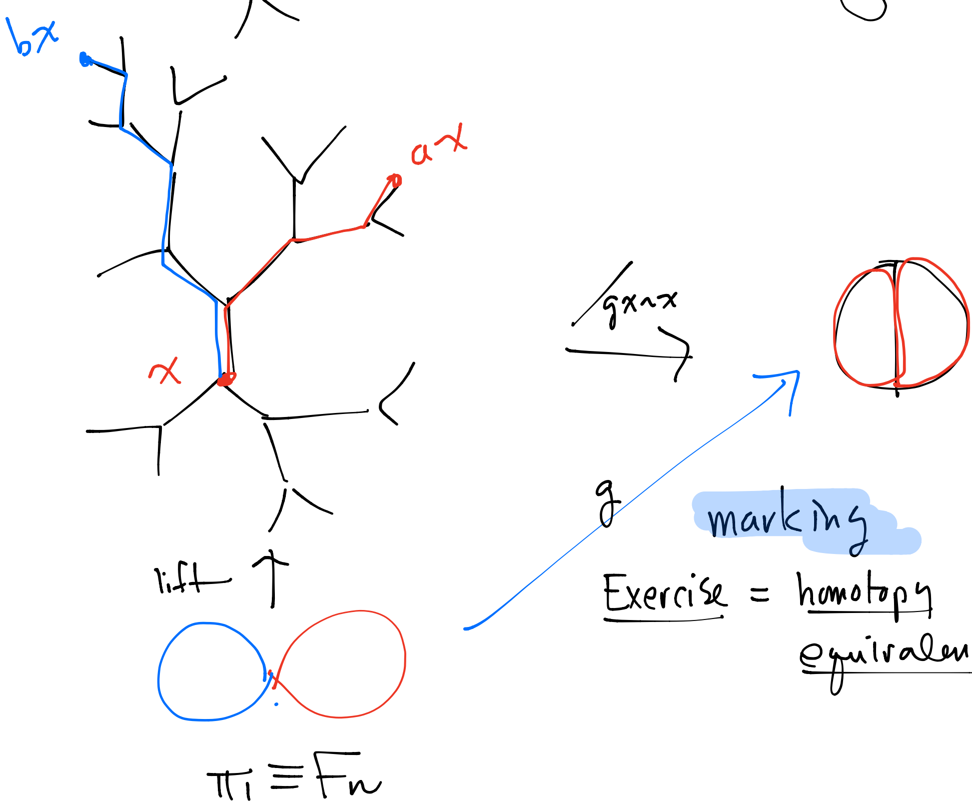
This definition of CV_n is succinct. Other advantages - generalizes to other classes of groups with nontrivial actions on trees (eg free products)

But it doesn't give much intuition about what CV_n <u>looks</u> like

Definition in terms of graphs

F_n acts freely on $T \Rightarrow$ quotient $T/_{gx\sim x}$ is a graph with $\pi_1 \cong F_n$





Exercise = homotopy equivalence

The <u>marking</u> identifies $\pi_1 R_n \equiv F_n$ with $\pi_1 G$

<u>Metric</u> on T descends to a metric on G

An <u>equivariant isometry</u> $T \to T'$ descends to an isometry $G \to G'$

<u>Exercise</u> Action on T is <u>minimal</u> $\Leftrightarrow$ G is finite and has no univalent vertices,

(and we can also ignore bivalent vertices)

Definition #2 Fix a rose R_n, identify $\pi_1 R_n \equiv F_n$

CV_n is the space of equivalence classes of marked metric graphs (G, g) where

- G is finite and has no univalent or bivalent vertices
- $g : R_n \longrightarrow G$ is a homotopy equivalence.

- $(G,g) \sim (G',g')$ if there is an isometry $G \to G'$ making
$$\begin{tikzcd} G \arrow[rr] & & G' \\ & R_n \arrow[ul, "g"] \arrow[ur, "g'"'] & \end{tikzcd}$$
commute up to homotopy

Action of $Out(F_n)$: Represent $\varphi \in Out(F_n)$ by $f: R_n \to R_n$ a homotopy equivalence. Then $(G,g)\cdot\varphi = (G, g\circ f)$:
$$\begin{tikzcd} G & \\ R_n \arrow[u, "g"] & R_n \arrow[l, "f"'] \arrow[ul, dash, "g\circ f"'] \end{tikzcd}$$

$Aut(F_n)$ also acts, but we use $Out(F_n)$ because we haven't specified basepoints in graphs.

If we use basepointed graphs and markings, and insist that homotopy equivalences preserve the basepoint we get Auter space.

It is often convenient to <u>normalize</u> the metrics, ie assume $\sum_{e \in G} l(e) = 1$,

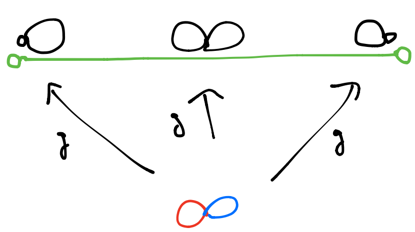
get CV_n (instead of cv_n)

Description in terms of graphs makes it easier to see local structure and topology, especially in CV_n.

CV_n is a union of open simplices

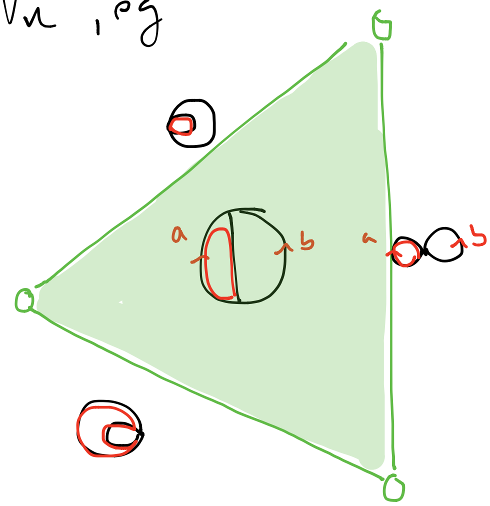
The simplex $\sigma(G,g)$ containing (G,g) consists of (G',g) : G' is obtained from G by varying edge lengths

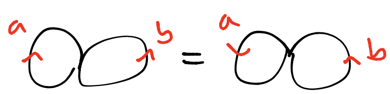
eg



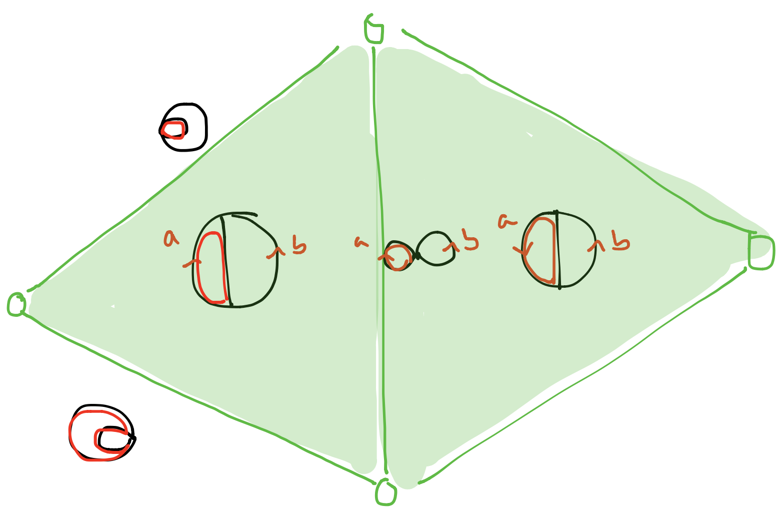
endpoints are missing, are not in CV_2

Some faces of a simplex are also simplices in CV_n, eg

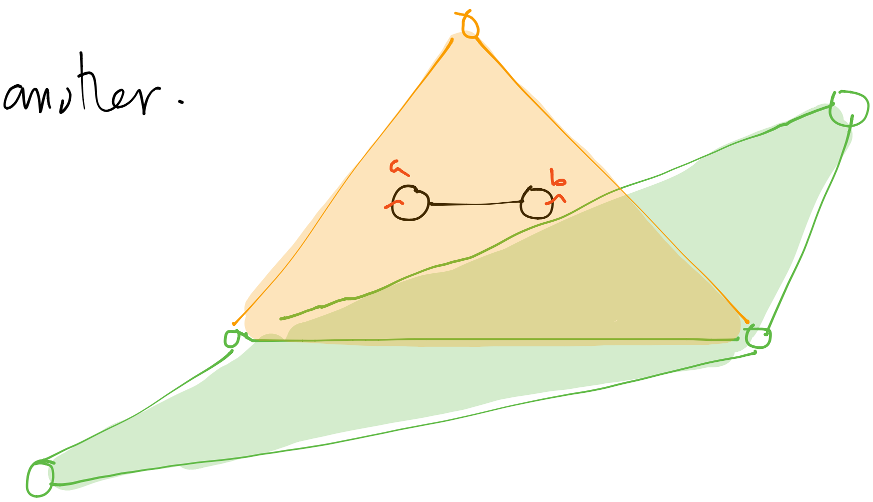


Since 

there's another simplex with the same face



and another.

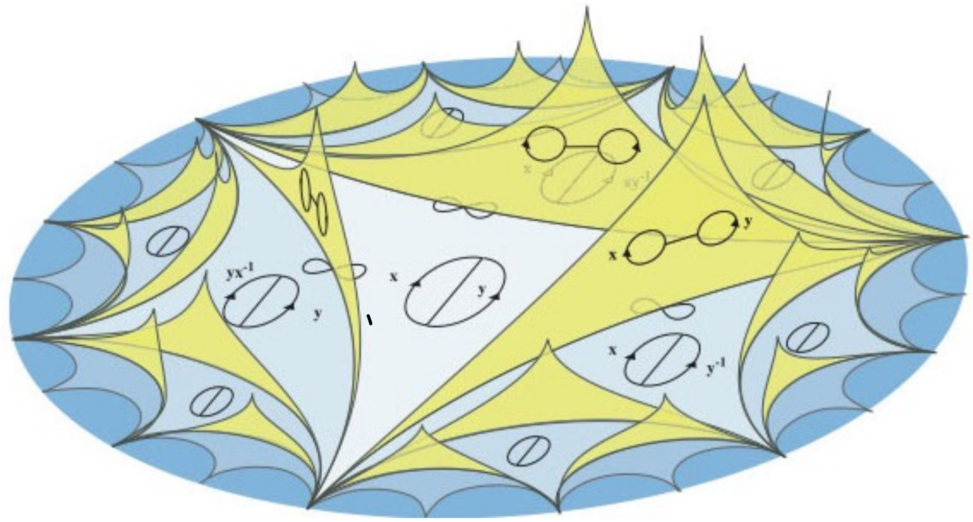


$$CV_n = \coprod \sigma(G,g) \Big/ \text{face identifications}$$

<u>Thm</u> (Paulin) The quotient topology is the same as the equivariant Gromov-Hausdorff topology. (possible reading project)

<u>Exercise</u>: $\dim CV_n = 3n-4$

(use $\chi(G) = 1-n$, valence$(v) \geq 3$)

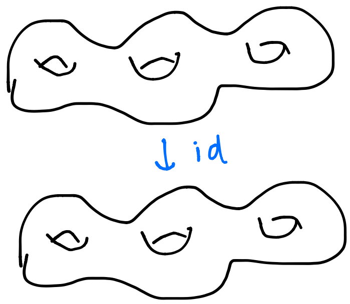


The "fins" seem unnecessary

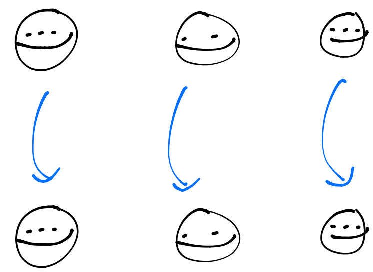
<u>Def</u> Reduced outer space $\overline{CV_n}$ is the subspace of (G, g) s.t. G has no separating edges.

There is an equivariant deformation retract $CV_n \to \overline{CV_n}$ given by shrinking all separating edges to points.

Third definition: Using $M_n = \#_n S^1 \times S^2$



or



A sphere system $\mathcal{S} \subset M_n$ is a collection of embedded 2-spheres $\mathcal{S} = \{s_1, \dots, s_k\}$ s.t.

- The s_i are disjoint
- No s_i bounds a ball
- No s_i is homotopic to any s_j

Sphere systems $\mathcal{S}, \mathcal{S}'$ are equivalent if there is an isotopy of M_n taking $\mathcal{S}$ to $\mathcal{S}'$

(Isotopy = homotopy $M_n \times I \to M_n$ s.t. each f_t is a homeomorphism)

A sphere system $\mathcal{S}$ is simple if all components of $M \setminus \mathcal{S}$ are simply connected (so are punctured balls)

<u>Def</u> A weighted sphere system is a sphere system $\mathcal{S} = \{s_1, \ldots, s_k\}$ s.t. each s_i is assigned a positive real weight λ_i.

Definition #3 CV_n is the space of weighted simple sphere systems in M_n

<u>Topology</u>: The set of <u>all</u> (unweighted) sphere systems forms a simplicial complex $S'(M_n)$

- vertices = sphere systems
- edge $\mathcal{S} - \mathcal{S}'$ if $\mathcal{S} \subset \mathcal{S}'$

  ie spheres in $\mathcal{S}$ are isotopic to spheres in $\mathcal{S}'$

- k-simplex $\leftrightarrow$ chain of k inclusions

  $\mathcal{S}_0 \subset \ldots \subset \mathcal{S}_k$

This is the barycentric subdivision of a simplicial complex which is even easier to describe

$S(M_n)$: vertices = isotopy classes of embedded 2-spheres

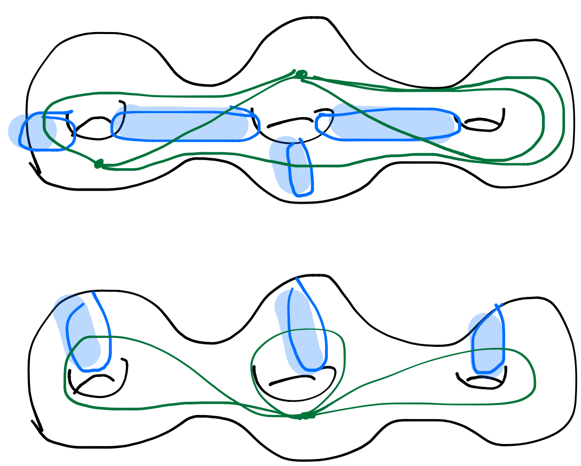
k-simplex = sphere system with k spheres

Put <u>weights</u> on spheres using barycentric coordinates. Then

$CV_n \subseteq S(M_n)$ is the subspace consisting of sphere systems with non-zero weights on a simple system.

Correspondence between simple sphere systems and marked metric graphs:

$\mathcal{S} \subset M_n$ cuts M_n into punctured balls



Take a vertex for each component, edge for each sphere to get a graph $G(\mathcal{S})$

Identify $\pi_1 M \equiv F_n$

The <u>marking</u> on $G(\mathcal{S})$ is given by

inclusion $\quad G(\mathcal{S}) \hookrightarrow M_n$

induces $\quad \pi_1(G(\mathcal{S})) \xrightarrow{\cong} \pi_1 M \equiv F_n$

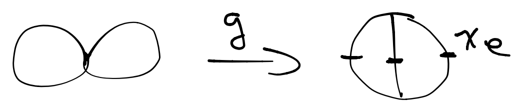
(ok, so take a homotopy inverse $M_n \to G(\mathcal{S}) \ldots$)

The <u>weights</u> on the $s_i \in S$ give lengths to the edges of $G(\mathcal{S})$.

Notice $G(\mathcal{S})$ has no univalent or bivalent vertices (s_i doesn't bound a ball, s_i, s_j not parallel).

The other way: Given (G, g) a marked graph, how do you get a sphere system in M_n?

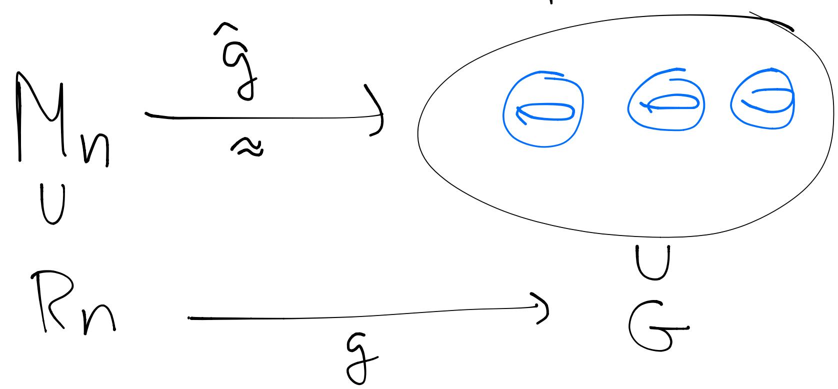
$R_n \xrightarrow{g} G$



- Take a pt x_e in each edge of G
- Fatten up R_n and G into handlebodies,

Each x_e becomes a disk D_e

ow double. Each D_e becomes a sphere S_e



Choose a diffeomorphism $\hat{g}$ realizing g on π_1

Then $S(G,g) = \{ \hat{g}^{-1}(S_{e_i}) \}$

Details are in <u>Hatcher</u>: Homology stability for $\mathrm{Out}(F_n)$

Action of $\mathrm{Out}\, F_n$: We saw any $\varphi \in \mathrm{Out}\, F_n$ can be realized by a diffeomorphism of M_n. This gives a map

$$\pi_0 \mathrm{Diff}(M_n) \longrightarrow \mathrm{Out}(F_n) \to 0$$

$\pi_0(\mathrm{Diff}\, M_n)$ obviously acts on sphere systems. To get $\mathrm{Out}(F_n)$ to act, we need to show the kernel acts trivially.

Recall Laudenbach's theorem:

Kernel is generated by Dehn twists in 2-spheres. So suffices to notice that a Dehn twist acts trivially on a sphere system: